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On the Construction of the Taub–NUT Congruence[1]

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Abstract

We describe how the null congruence tangent to the multiple Debever–Penrose direction of the Taub–NUT solution of Einstein's vacuum field equations may be constructed emanating into the future from a timelike world tube having normal cross sections. The curious shapes of the cross sections of the world tube are plotted, using a computer, for critical ranges of the parameter b/R_0 where b is the Taub–NUT parameter and R_0 is the "radius" of the world tube. It is found that these cross sections can be maintained spatially compact only for some values of b/R_0.

1. *Introduction*

The Taub–NUT (Taub, 1951; Newman, Unti, and Tamburino, 1963) solution of Einstein's vacuum field equations may be written in the form (cf. Robinson & Trautman, 1964)

$$ds^2 = (R^2 + b^2)(d\theta^2 + \sin^2\theta\, d\phi^2) - 2\, dR\, d\Sigma - f\, d\Sigma^2 \tag{1.1}$$

where

$$d\Sigma = k_i\, dx^i = -du + 4b \sin^2 \tfrac{1}{2}\theta\, d\phi \tag{1.2}$$

$$f = 1 - \frac{2mR + b^2}{R^2 + b^2} \tag{1.3}$$

Here m and b are constants and $(x^1, x^2, x^3, x^4) \equiv (\theta, \phi, R, u)$. The solution is of Petrov type D with $k^i = \partial x^i/\partial R$ as multiple Debever–Penrose null direction The null, geodesic, and shear-free integral curves of k^i have expansion and twist given by

$$\rho = (R + ib)^{-1} = (\text{expansion}) + i\,(\text{twist}) \tag{1.4}$$

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The solution has been extensively studied by Misner (1963; 1967). It appears to be mainly of geometrical interest and to have little to do with reality. Nevertheless there have been some ingenious physical interpretations suggested for it (Demianski & Newman, 1966; Bonnor, 1969; Dowker & Roche, 1967; Dowker, 1974).

For large values of R, Eq. (1.1) approaches the Eddington form of the Schwarzschild solution. This may be achieved, of course, by taking $b = 0$. The line element (1.1) becomes then

$$ds^2 = (R^2 + b^2)(d\theta^2 + \sin^2\theta\, d\phi^2) - 2\, du\, dR - f\, du^2 \tag{1.5}$$

where

$$f = 1 - 2m/R \tag{1.6}$$

In this case $d\Sigma = k_i dx^i = -du$, $\rho = R^{-1}$ so that k_i is hyper-surface-orthogonal. A "background" Minkowskian space-time is obtained from (1.5) by putting $m = 0$. In this background the null congruence tangent to k^i may be constructed (cf. Hogan, 1975). This is achieved by first choosing, in the background, a timelike world tube $R = R_0 > 0$ (R_0 constant) having covariantly constant generators $\lambda^i = \partial x^i/\partial u$ (with u proper-time along them with respect to the metric of the Minkowskian background), and spherical normal sections. Then k^i points out into the future from every event on the world tube, the direction of k^i being in the two-space spanned by the unit tangent to the generator, λ^i, and the unit normal to the world tube, at each event on the world tube. Hence $k_i\, dx^i = \lambda_i\, dx^i + dR$.

We now inquire to what extent can we apply such a construction to the Taub–NUT congruence k^i given in (1.2)? Can we construct the null congruence (1.2) emanating out into the future from a timelike world tube having covariantly constant generators and spatially compact normal sections? We find that this *can* be done *only* for a certain range of values of b/R_0. The world tube has very curious normal cross sections, and we display computer-generated plots of some of them. We obtain the line element of the space-time containing the world tube. It is *not* Minkowskian, nor does the Ricci tensor vanish. It is not asymptotically flat, but it is, however, singularity-free [a property it shares with the line element (1.1), cf. Misner, 1967].

2. *Construction of the Null Congruence*

Let V_4 be a space-time containing a timelike world tube. The world tube is generated by a two-parameter family of timelike world lines $C(\theta, \phi)$: $x^i = x^i(u; \theta, \phi)$ where $-\infty < u < +\infty$ and θ, ϕ are polar coordinates labeling the generators with $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$. Then if $\lambda^i = \partial x^i/\partial u$ we take $\lambda_i \lambda^i = -1$. Let $x^i = x^i(R)$, $R_0 \leqslant R < +\infty$ be a congruence of shear-free, null geodesics emanating into the future from every event on the world tube, with $k^i = \partial x^i/\partial R$. If the generators λ^i are covariantly constant and if the normal sections of the world tube, i.e., the two-surfaces $R = R_0$, $u = \text{const}$, have axial symmetry and become spherical as $R_0 \to +\infty$ (this will be sufficiently general for our purposes) then Hogan has shown that the line element of the space-time V_4 may be written

$$ds^2 = (R^2 + b^2)(d\theta^2 + \sin^2\theta\, d\phi^2) + (2\,dR + du + k\,d\phi)(-du + k\,d\phi) \tag{2.1}$$

where $b = b(\theta)$, $k = k(\theta)$,

$$dk/d\theta = 2b\sin\theta \tag{2.2}$$

and the shear-free, null geodesic congruence is

$$d\Sigma = k_i\,dx^i = -du + k\,d\phi \tag{2.3}$$

The congruence has expansion and twist given by

$$\rho = (R + ib)^{-1} \tag{2.4}$$

It may also be shown (cf. Hogan, 1975) that if, in addition, the Riemann tensor vanishes for the line element (2.1) then this, together with (2.2), determines uniquely the unknown functions b and k. They are found to be

$$b = a\cos\theta, \quad k = a\sin^2\theta \qquad (a = \text{const}) \tag{2.5}$$

and so the congruence k^i becomes the multiple Debever-Penrose direction of the Kerr solution (Kerr, 1963). In this paper we are interested in the special case of b = const. We may now integrate (2.2) to obtain

$$k = 4b\sin^2\tfrac{1}{2}\theta \tag{2.6}$$

where we have chosen the constant of integration to be $2b$, for simplicity. We now have a line element

$$ds^2 = (R^2 + b^2)(d\theta^2 + \sin^2\theta\, d\phi^2) + (2\,dR + du + 4b\sin^2\tfrac{1}{2}\theta\, d\phi)$$
$$\times\,(-du + 4b\sin^2\tfrac{1}{2}\theta\, d\theta) \tag{2.7}$$

with respect to which

$$d\Sigma = -\,du + 4b\sin^2\tfrac{1}{2}\theta\, d\phi \tag{2.8}$$

is a shear-free, null geodesic congruence having expansion and twist given by (2.4) with b = const. This coincides with the null congruence (1.2) associated with the Taub-NUT solution (1.1).

3. *The Normal Cross Sections of the World Tube*

The line element for the normal cross sections of the world tube $R = R_0$ which generate the congruence (2.8) is obtained by taking u = const, $R = R_0$ in (2.7). It is

$$dl^2 = (R_0^2 + b^2)\,d\theta^2 + [(R_0^2 + b^2)\sin^2\theta + 16b^2\sin^4\tfrac{1}{2}\theta]\,d\phi^2 \tag{3.1}$$

The two-surfaces described by this line element are axially symmetric and for certain ranges of values of b/R_0 may be numerically embedded in a

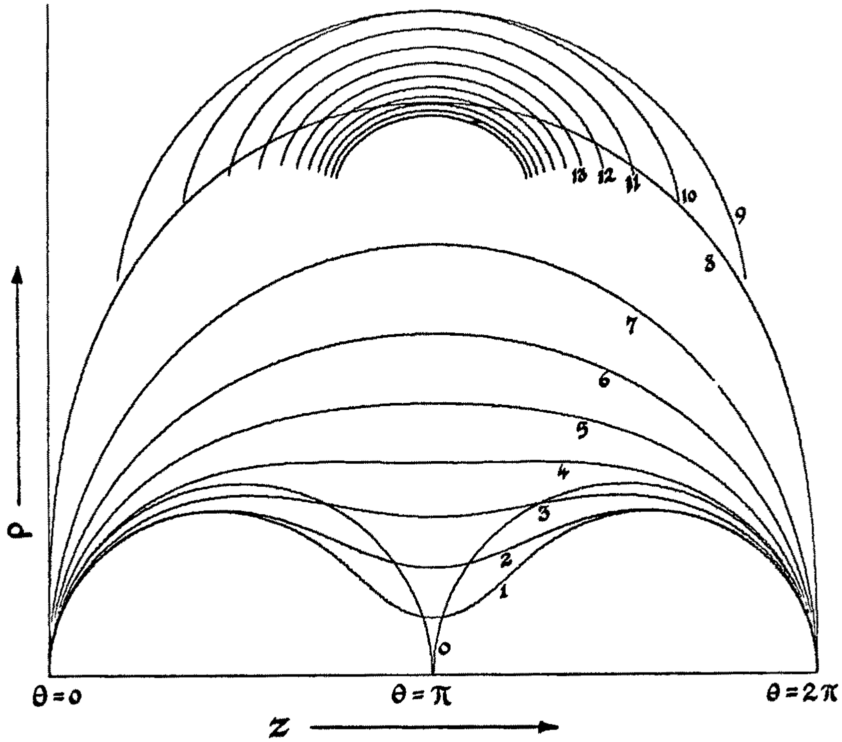

Figure 1—ϕ = const slices of the embedded two-surfaces with line element dl^2 given in (2.1). In this figure $dz = \sqrt{dl^2 - d\rho^2}$, where $\rho = [(R_0{}^2 + b^2)\sin^2\theta + 16b^2\sin^4\frac{1}{2}\theta]^{1/2}$. The numbers on each curve refer to the value of b/R_0 in units of $(8\sqrt{2})^{-1}$.

Euclidean three-space. For $b \neq 0$ these surfaces are not closed. However, they may be closed if the range of θ is increased to $[0, 2\pi]$. They have reflectional symmetry through the plane $\theta = \pi$ with a smooth joining at $\theta = \pi$.

Figure 1 shows the ϕ = const slices for a family of surfaces with b/R_0 taking values in the range [0, 1.68] approximately. For values of $b/R_0 > (\sqrt{2})$ portions of the surface may not be embedded, since the quantity appearing under the radical sign in the expression for dz becomes negative. In this case even the smooth extension of the range of θ to 2π does not result in real closed two-surfaces. For large values of $b/R_0 \sim 10^6$ only that portion of the surface with $\theta \in [(0.80678 \pm 0.00002)\pi, (1.19322 \pm 0.00002)\pi]$ may be embedded. For $b/R_0 > 10$ these surfaces approach a uniform shape.

Figure 2 shows the full two-surfaces for the limiting cases $b/R_0 \sim 10^6$ and $b/R_0 \sim 0.05$. In the latter case the surface can be seen to approach two smoothly touching spheres which pinch off as $b \to 0$.

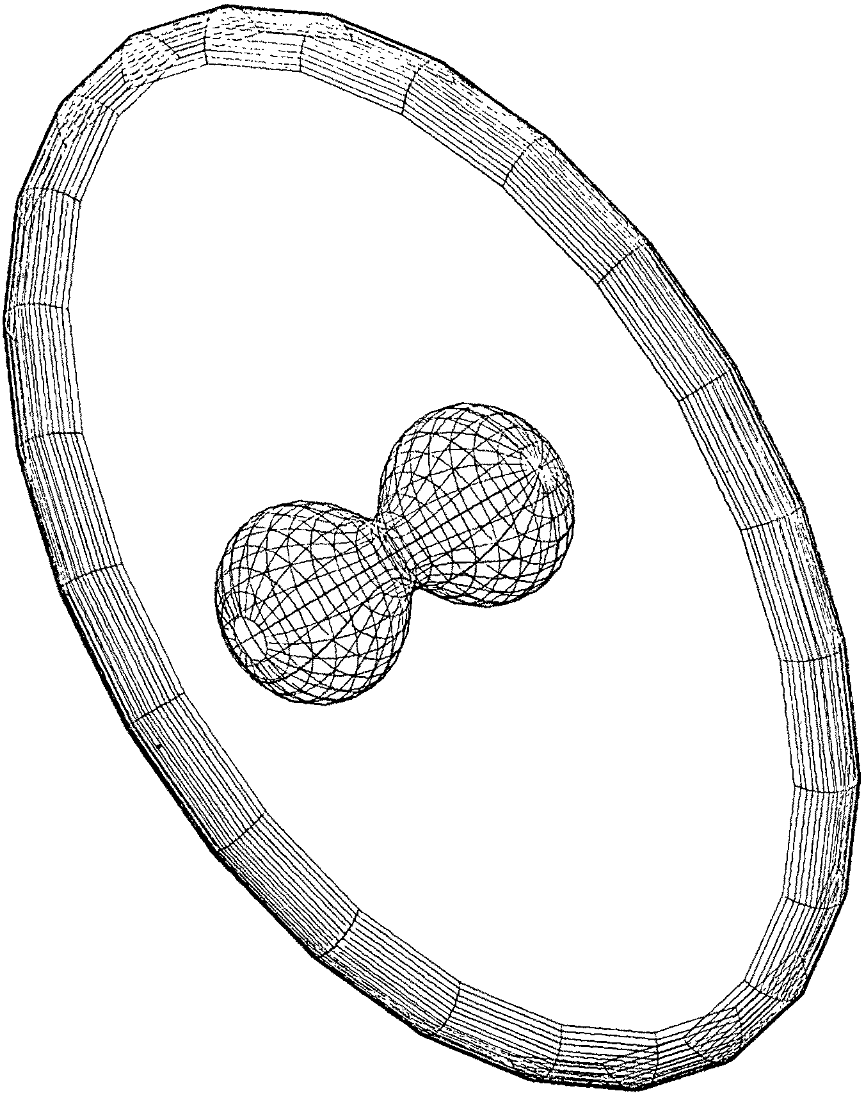

Figure 2–The two-surfaces for the limiting cases $b/R_0 \sim 10^6$ (outer surface) and $b/R_0 \sim 0.05$ (inner surface).

4. *Discussion*

We have obtained a line element of a Riemannian four-space which has associated with it a twisting, diverging null geodesic congruence having the properties of the null congruence associated with the Taub-NUT solution of

Einstein's vacuum field equations. The congruence emanates into the future from a timelike world tube, having covariantly constant generators and normal cross sections which must be specially chosen to reveal the null congruence. These sections are not, in general, closed two-surfaces, but they may be closed smoothly by extending the range of one of the coordinates. However, if $b/R_0 > (\sqrt{2})^{-1}$ even this extension does not result in real closed surfaces. In the limit $R_0 \to +\infty$ the cross sections are spheres, hence we must "identify" corresponding points on the original noncompact two-surfaces (with $\theta \in [0, \pi]$) and their extensions (with $\theta \in [\pi, 2\pi]$) by reflection through the plane $\theta = \pi$, otherwise we see from the figures that we obtain *two* spheres smoothly touching for $R_0 \gg b$. We have presented computer-generated plots of these two-surfaces for interesting values of the parameter b/R_0.

Finally we point out that the line element (2.7) does not have a vanishing Riemann tensor or Ricci tensor, it is not asymptotically flat (in the limit $R \to \infty$), and it is singularity-free. Its appeal lies in its curious relationship to the Taub-NUT solution which we have demonstrated.

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